J. Math. Anal. Appl. 376 (2011) 365–372

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Approximation of the tail probability of randomly weighted sums of dependent random variables with dominated variation

Lan Yi, Yu Chen *, Chun Su

Department of Statistics and Finance, University of Science and Technology of China, Hefei Anhui, 230026, China

ARTICLE INFO

Article history:
Received 7 January 2010
Available online 14 October 2010
Submitted by V. Pozdnyakov

Keywords:
Asymptotic
Randomly weighted sum
Dominatedly-varying tail
Tail probability
Quasi-asymptotic independence

ABSTRACT

This paper deals with the approximation of the tail probability of randomly weighted sums of a sequence of pairwise quasi-asymptotically independent but non-identically distributed dominatedly-varying-tailed random variables. The weights are independent of the former sequence, satisfying some assumptions about the moments. But no requirements on the dependence structure of the weights are imposed.

1. Introduction

In this paper, we study the asymptotic tail probabilities of randomly weighted sums of some real-valued dominatedly-varying-tailed random variable sequence $\{X_i,\ i=1,2,\ldots\}$ with nonnegative random weight sequence $\{\Theta_i,\ i=1,2,\ldots\}$, which is independent of $\{X_i,\ i=1,2,\ldots\}$. Denote the randomly weighted sums and their maxima as follows:

$$S_n^\theta=\sum_{i=1}^n \Theta_i X_i,\quad n\geqslant 1,\qquad M_n^\theta=\max_{1\leqslant m\leqslant n} S_m^\theta,\quad n\geqslant 1,$$
$$S_\infty^\theta=\sum_{i=1}^\infty \Theta_i X_i,\qquad S_\infty^{+\theta}=\sum_{i=1}^\infty \Theta_i X_i^+,\qquad M_\infty^\theta=\max_{1\leqslant m<\infty} S_m^\theta,$$

where $x^+=\max\{x,0\}=x\vee 0$ and $x^-=-\min\{x,0\}=-(x\wedge 0)$ for any real number x. From now on, we assume that the weights $\{\Theta_i,\ i=1,2,\ldots\}$ are not degenerate at 0 (to avoid triviality).

Estimating the asymptotic tail probabilities of randomly weighted sums is quite common in insurance and financial risk models, where the two random variable series have concrete meanings. Take the discrete-time risk model for example. In such models, when considering the asset risk of the surplus of one insurance company, we represent the net loss (the total claim amount minus the total incoming premium) of the insurer at year i by X_i and represent the discount factor since year i by Θ_i. Then the randomly weighted sum S_n^θ represents the total discounted amount of the net loss till the end of year n. In such an environment, $\Pr(S_n^\theta>x\colon$ for some $1\leqslant n<\infty)$ or $\Pr(M_\infty^\theta>x)$ is the ultimate ruin probability;

* Corresponding author.
E-mail addresses: debbie32@mail.ustc.edu.cn (L. Yi), cyu@ustc.edu.cn (Y. Chen), suchun@ustc.edu.cn (C. Su).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.10.020

$\Pr(S_n^\theta > x\text{: for some } 1 \leqslant n \leqslant T)$ or $\Pr(M_T^\theta > x)$ is the ruin probability within finite horizon T, where x represents the initial surplus of the insurance company. More information about the discrete-time risk model and the finite and infinite ruin probabilities can be found in many articles, for example: Nyrhinen [13,14], and Tang [16,17].

Due to the importance of this topic, there are many decent papers addressing it. As far as we know, Resnick and Willekens [15] got the relation

$$\Pr\left(\sum_{i=1}^{\infty} \Theta_i X_i > x\right) \sim \sum_{i=1}^{\infty} \Pr(\Theta_i X_i > x) \sim \overline{F}(x) \sum_{i=1}^{\infty} E\Theta_i^{\alpha} \tag{1}$$

where $\overline{F}(x) = 1 - F(x)$, for nonnegative independent and identically distributed (i.i.d.) random variables (r.v.'s) $\{X_i,\ i = 1, 2, \ldots\}$ with common distribution F from the regularly-varying-tailed class $\mathcal{R}_{-\alpha}$, $\alpha > 0$ (definition is in (4)), when the related sums of the moments of the weights are finite.

Later Tang and Tsitsiashvili [18] extended similar relations to i.i.d. real-valued r.v.'s $\{X_i,\ i = 1, 2, \ldots\}$ from the class of $\mathcal{L} \cap \mathcal{D}$ (definitions are in (2) and (3)) with special weights $\Theta_i = \prod_{k=1}^{i} Y_k$, $i = 1, 2, \ldots,$ where $\{Y_k,\ k = 1, 2, \ldots\}$ are i.i.d. nonnegative r.v.'s and have some finite moments. The uniformity of the asymptotic relation therein was verified in Tang and Tsitsiashvili [20] for the ERV class (definition is in (5)). Chen and Su [5] treated the same problem by making assumptions about the density functions.

Tang and Tsitsiashvili [19] again established similar relations for i.i.d. real-valued r.v.'s $\{X_i,\ i = 1, 2, \ldots\}$ from several heavy-tailed distribution classes with uniformly bounded weights. Chen et al. [4] derived the relation for M_n^θ of i.i.d. real-valued r.v.'s $\{X_i,\ i = 1, 2, \ldots\}$ from the regularly-varying-tailed class $\mathcal{R}_{-\alpha}$, where the weights are bounded above with bounds satisfying some property.

Goovaerts et al. [12] extended the results in Resnick and Willekens [15] to real-valued situation for M_n^θ. Wang et al. [21] extended them to i.i.d. real-valued r.v.'s $\{X_i,\ i = 1, 2, \ldots\}$ from the consistently-varying-tailed class $\mathcal{C}$ (definition is in (6)) for M_n^θ. And Wang and Tang [22] further extended them to the class of $\mathcal{L} \cap \mathcal{D}$. Some other results about weighted sums may be found in Geluk and De Vries [10] and others.

So far, $\{X_i,\ i = 1, 2, \ldots\}$ are supposed to be independent. Though such assumption facilitates mathematical treatment, independence is too stringent in application. Thus, more and more attention is paid to dependent situations. As far as we know, when deriving the relation $\Pr(\sum_{k=1}^{n} X_k > x) \sim \sum_{k=1}^{n} \overline{F}_k(x)$ for independent but not identically-distributed real-valued random variables $X_1, \ldots, X_n$, Geluk and Tang [11] introduced two quite general dependence structures, called Assumption A and Assumption B. Zhang et al. [24] introduced bivariate upper tail independence structure from a bivariate copula function and derived relation (1) for real-valued r.v.'s $\{X_i,\ i = 1, 2, \ldots\}$ with one common distribution function from the ERV class. Chen and Yuen [6] adopted the pairwise quasi-asymptotic independence structure for nonnegative r.v.'s $\{X_i,\ i = 1, 2, \ldots\}$ from the ERV class and reached a similar relation to (1). Gao and Wang [9] extended the results in Zhang et al. [24] to dominatedly-varying-tailed class, which was studied in Aljančić and Arandelović [1].

Inspiring by the results in Chen and Yuen [6], we are going to study the case when $\{X_i,\ i = 1, 2, \ldots\}$ are non-identically distributed and with pairwise quasi-asymptotic independence structure.

The remaining part of this paper is divided into three sections. In Section 2 we prepare some related knowledge for use and also explain some symbols. In Section 3 we state our main results and their proofs are presented in Section 4.

2. Preliminaries

The usual assumption about the distribution function (d.f.) F of X is F is heavy-tailed. We say a r.v. X or a d.f. F belongs to the class of heavy-tailed distributions, if $Ee^{tX} = \infty$, $\forall t > 0$. Here we review some important classes of heavy-tailed distributions.

We say that a d.f. F is long-tailed, denoted as $F \in \mathcal{L}$, if for any $y > 0$, we have

$$\lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = 1. \tag{2}$$

We say that a d.f. F belongs to the dominatedly-varying-tailed class, denoted as $F \in \mathcal{D}$, if for any $y > 0$, we have

$$\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty. \tag{3}$$

We say that a d.f. F belongs to the regularly-varying-tailed class, denoted as $F \in \mathcal{R}_{-\alpha}$, if there exists some $\alpha \geqslant 0$, such that for any $y > 0$, we have

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha}. \tag{4}$$

We say that a d.f. F belongs to the extended-regularly-varying-tailed class, denoted as $F \in ERV(-\alpha, -\beta)$, if there exist some $0 \leqslant \alpha \leqslant \beta < \infty$ such that for any $y > 1$, we have

$$y^{-\beta} \leqslant \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \leqslant \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \leqslant y^{-\alpha}. \tag{5}$$

We say that a d.f. F belongs to the consistently-varying-tailed class, denoted as $F \in \mathcal{C}$, if

$$\lim_{\mu\nearrow 1} \limsup_{x\to\infty} \frac{\overline{F}(\mu x)}{\overline{F}(x)} = 1 \quad \text{or} \quad \lim_{\mu\searrow 1} \liminf_{x\to\infty} \frac{\overline{F}(\mu x)}{\overline{F}(x)} = 1. \tag{6}$$

The following relationships are well known:

$$\mathcal{R} \subset ERV \subset \mathcal{C} \subset \mathcal{D} \cap \mathcal{L} \subset \mathcal{L}.$$

For more details about the classes of heavy-tailed distributions, please refer to Bingham et al. [2] and Embrechts et al. [8]. For some examples to illustrate the relationships between those classes, Cai and Tang [3] and Cline and Samorodnitsky [7] are recommended additionally.

Now let's introduce the upper and lower Matuszewska indices ($\mathbb{J}_F^+$ and $\mathbb{J}_F^-$) for distribution F. We adopt the definition in Tang and Tsitsiashvili [18]:

$$\mathbb{J}_F^+ = -\lim_{y\to\infty} \frac{\log \overline{F}_*(y)}{\log y}, \qquad \mathbb{J}_F^- = -\lim_{y\to\infty} \frac{\log \overline{F}^*(y)}{\log y},$$

where $\overline{F}_*(y) = \liminf_{x\to\infty} \overline{F}(xy)/\overline{F}(x)$ and $\overline{F}^*(y) = \limsup_{x\to\infty} \overline{F}(xy)/\overline{F}(x)$. These indices were introduced in Bingham et al. [2] first, and later Cline and Samorodnitsky [7] restudied them. But for our purpose, the way in Tang and Tsitsiashvili [18] is more convenient. By the way, due to Theorem 2.1.5 in Bingham et al. [2], it's easy to get the following equalities:

$$\mathbb{J}_F^+ = \inf\left\{-\frac{\log \overline{F}_*(y)}{\log y}\colon\ y > 1\right\}, \qquad \mathbb{J}_F^- = \sup\left\{-\frac{\log \overline{F}^*(y)}{\log y}\colon\ y > 1\right\}.$$

Now we introduce another index for distribution F as in Yang and Wang [23]:

$$L_F = \lim_{y\downarrow 1} \overline{F}_*(y). \tag{7}$$

From the definition of $\overline{F}_*(y)$ and $\overline{F}^*(y)$, we know: for any $y > 0$, $\overline{F}^*(y) = 1/\overline{F}_*(\frac{1}{y})$. Thus, we have $L_F = \lim_{y\downarrow 1} \overline{F}_*(y) = 1/\lim_{y\uparrow 1} \overline{F}^*(y)$, which will be applied often later. By the definition of the class $\mathcal{C}$, it is easy to see $L_F = 1$ for $F \in \mathcal{C}$.

Now let's introduce the definition of the dependence structure of random variables we will adopt: quasi-asymptotic independence. This is defined in Chen and Yuen [6].

Definition 1 *(Quasi-asymptotic independence).* Two nonnegative random variables X_1 and X_2, with distributions F_1 and F_2 respectively, are said to be quasi-asymptotically independent if

$$\lim_{x\to\infty} \frac{\Pr(X_1 > x, X_2 > x)}{\Pr(X_1 > x) + \Pr(X_2 > x)} = 0. \tag{8}$$

More generally, two real-valued random variables, X_1 and X_2, are still said to be quasi-asymptotically independent if the relation (8) holds with (X_1, X_2) in the numerator replaced by $\{X_1^+, X_2^+\}$, $\{X_1^+, X_2^-\}$, $\{X_1^-, X_2^+\}$.

In fact random variables X_1 and X_2 with identical distribution are called to be asymptotically independent or upper tail independent if the relation (8) holds. See, for example, Zhang et al. [24].

We close this section by explaining some symbols which will be used later. We will use $\lesssim$, $\gtrsim$, $\sim$ and $\asymp$ to connect two functions, say $f_1(x)$ and $f_2(x)$, as follows: $f_1(x) \lesssim f_2(x)$ when $\limsup_{x\to\infty} \frac{f_1(x)}{f_2(x)} \leqslant 1$; $f_1(x) \gtrsim f_2(x)$ when $\liminf_{x\to\infty} \frac{f_1(x)}{f_2(x)} \geqslant 1$; $f_1(x) \sim f_2(x)$ when $\lim_{x\to\infty} \frac{f_1(x)}{f_2(x)} = 1$; $f_1(x) \asymp f_2(x)$ when $0 < \liminf_{x\to\infty} \frac{f_1(x)}{f_2(x)} \leqslant \limsup_{x\to\infty} \frac{f_1(x)}{f_2(x)} < \infty$. We say $f_1(x)$ and $f_2(x)$ are weakly equivalent if $f_1(x) \asymp f_2(x)$.

3. Main results and some remarks

Suppose that $\{X_i,\ i = 1, 2, \ldots\}$ are pairwise quasi-asymptotically independent but non-identically-distributed real-valued r.v.'s, denoting the distribution of X_i by $F_i \in \mathcal{D}$, $i \geqslant 1$. The nonnegative sequence of $\{\Theta_i,\ i = 1, 2, \ldots\}$ is independent of $\{X_i,\ i = 1, 2, \ldots\}$.

Assumptions. For convenience, we state the following assumptions:

(A1) $\{X_i,\ i = 1, 2, \ldots\}$ satisfy

$$\lim_{x\to\infty} \frac{\Pr(X_i < -x)}{\Pr(X_i > x)} = 0, \quad i \geqslant 1.$$

(A2) The tails of the distribution functions $F_i(x)$, $i \geqslant 1$, satisfy the following relation:

$$0 < S := \inf_{i\geqslant 1} \liminf_{x\to\infty} \frac{\overline{F}_i(x)}{\overline{G}(x)} \leqslant \sup_{i\geqslant 1} \limsup_{x\to\infty} \frac{\overline{F}_i(x)}{\overline{G}(x)} =: M < \infty, \tag{9}$$

where G is some distribution function.

Obviously, G is dominatedly-varying-tailed, and by the definitions of upper and lower Matuszewska indices it is not difficult to see that $\mathbb{J}_G^{\pm} = \mathbb{J}_{F_i}^{\pm}$. And it is also easy to see $L_{F_i} \geqslant \frac{S}{M} L_G$, $i \geqslant 1$, and $\bigwedge_i L_{F_i} := \lim_{n\to\infty} \bigwedge_{1\leqslant i\leqslant n} L_{F_i} = \lim_{y\downarrow 1} \bigwedge_i \overline{F}_{i*}(y)$. We introduce the following constant for later use:

$$L := \bigwedge_i L_{F_i}. \tag{10}$$

Due to assumption (A2), by Potter's inequality in Bingham et al. [2], for $p_1 < \mathbb{J}_G^-$, $p_2 > \mathbb{J}_G^+$, there exist positive and finite constants C_1, D_1, C_2, D_2, such that for all k,

$$\frac{\overline{F}_k(y)}{\overline{F}_k(x)} \geqslant C_1 \left(\frac{x}{y}\right)^{p_1}, \quad \text{for all } x \geqslant y \geqslant D_1, \tag{11}$$

$$\frac{\overline{F}_k(y)}{\overline{F}_k(x)} \leqslant C_2 \left(\frac{x}{y}\right)^{p_2}, \quad \text{for all } x \geqslant y \geqslant D_2. \tag{12}$$

Here are our main results and their proofs are in Section 4.

Theorem 1. *Suppose assumption* (A1) *holds. For some fixed n, let* $E\Theta_i^p < \infty$ $(1 \leqslant i \leqslant n)$, *for some* $p > \bigvee_{j=1}^n \mathbb{J}_{F_j}^+$. *Then we have*

$$L_n \sum_{i=1}^n \Pr(\Theta_i X_i > x) \lesssim \Pr\big(S_n^\theta > x\big) \leqslant \Pr\big(M_n^\theta > x\big) \lesssim L_n^{-1} \sum_{i=1}^n \Pr(\Theta_i X_i > x),$$

$$\sum_{i=1}^n \Pr(\Theta_i X_i > x) \lesssim \Pr\left(\sum_{i=1}^n \Theta_i X_i^+ > x\right) \lesssim L_n^{-1} \sum_{i=1}^n \Pr(\Theta_i X_i > x),$$

where $L_n = \bigwedge_{1\leqslant k\leqslant n} L_{F_k}$ *and* L_F *is defined in* (7).

Remark 1. (1) If additionally $F_i \in \mathcal{C}$, then $L_{F_i} = 1$, $1 \leqslant i \leqslant n$. Thus by Theorem 1 we have

$$\Pr\left(\sum_{i=1}^n \Theta_i X_i > x\right) \sim \sum_{i=1}^n \Pr(\Theta_i X_i > x),$$

which is Theorem 3.2 in Chen and Yuen [6].

(2) If additionally assumption (A2) holds, we have for all n:

$$L \sum_{i=1}^n \Pr(\Theta_i X_i > x) \lesssim \Pr\big(S_n^\theta > x\big) \leqslant \Pr\big(M_n^\theta > x\big) \lesssim L^{-1} \sum_{i=1}^n \Pr(\Theta_i X_i > x),$$

$$\sum_{i=1}^n \Pr(\Theta_i X_i > x) \lesssim \Pr\left(\sum_{i=1}^n \Theta_i X_i^+ > x\right) \lesssim L^{-1} \sum_{i=1}^n \Pr(\Theta_i X_i > x),$$

where L is defined in (10).

Theorem 2. *Suppose both assumptions* (A1) *and* (A2) *hold, and* $\mathbb{J}_G^- > 0$. *Let either of the following statements about* Θ_i*'s hold*:

(1) *if* $0 < \mathbb{J}_G^+ < 1$, *there exists* $\delta > 0$ *such that* $\mathbb{J}_G^- - \delta =: p_1 > 0$, $\mathbb{J}_G^+ + \delta =: p_2 < 1$ *and*

$$\sum_{i=1}^{\infty} E\Theta_i^{p_1} < \infty, \qquad \sum_{i=1}^{\infty} E\Theta_i^{p_2} < \infty, \tag{13}$$

(2) *if* $1 \leqslant \mathbb{J}_G^+ < \infty$, *there exists* $\delta > 0$ *such that* $\mathbb{J}_G^- - \delta =: p_1 > 0$ (*set* $\mathbb{J}_G^+ + \delta =: p_2$), *and*

$$\sum_{i=1}^{\infty} \left(E\Theta_i^{p_1}\right)^{\frac{1}{p_2}} < \infty, \qquad \sum_{i=1}^{\infty} \left(E\Theta_i^{p_2}\right)^{\frac{1}{p_2}} < \infty. \tag{14}$$

Then we have

$$L\sum_{i=1}^{\infty} \Pr(\Theta_i X_i > x) \lesssim \Pr\left(M_\infty^\theta > x\right) \lesssim L^{-2}\sum_{i=1}^{\infty} \Pr(\Theta_i X_i > x), \tag{15}$$

$$\sum_{i=1}^{\infty} \Pr(\Theta_i X_i > x) \lesssim \Pr\left(S_\infty^{+\theta} > x\right) \lesssim L^{-2}\sum_{i=1}^{\infty} \Pr(\Theta_i X_i > x). \tag{16}$$

Remark 2. If $\{X_1, X_2, \ldots\}$ are identically distributed by F, the relation (9) in assumption (A2) holds with $S = M = 1$. Therefore $L = L_F$, then our results reduces to those in Gao and Wang [9]. And if $F \in \mathcal{C}$, then $L_F = 1$ and our results reduces to those in Chen and Yuen [6]. Thus Theorem 2 has extended the results in Gao and Wang [9] and Chen and Yuen [6].

4. Proofs of main results

4.1. Some lemmas

Hereafter, C denotes an absolute and positive constant whose value may vary from place to place.

Lemma 1. *Let* X *and* Θ *be two independent random variables.* X *is distributed by* $F \in \mathcal{D}$ *with Matuszewska indices* $0 < \mathbb{J}_F^- \leqslant \mathbb{J}_F^+ < \infty$; Θ *is nonnegative. Then for any fixed* p_1, p_2 *satisfying* $0 < p_1 < \mathbb{J}_F^- \leqslant \mathbb{J}_F^+ < p_2 < \infty$, *there exists some constant* C (*irrespective of* Θ) *such that, for all large* x,

$$\Pr(\Theta X > x) \leqslant C\overline{F}(x)\left(E\Theta^{p_1} \vee E\Theta^{p_2}\right). \tag{17}$$

Proof. We follow Wang and Tang [22] to prove this lemma. Choose D_2 as in the inequality (12). For $x \geqslant D_2$ we have:

$$\Pr(\Theta X > x) = \Pr(\Theta X > x, \Theta \geqslant x/D_2) + \Pr(\Theta X > x, x/D_2 > \Theta \geqslant 1) + \Pr(\Theta X > x, \Theta < 1) =: I_1 + I_2 + I_3.$$

According to (12), we have

$$I_1 \leqslant \Pr(\Theta \geqslant x/D_2) \leqslant D_2^{p_2} x^{-p_2} E\Theta^{p_2} \leqslant \frac{C_2}{\overline{F}(D_2)}\overline{F}(x)E\Theta^{p_2}.$$

To deal with I_2, we introduce V to denote the distribution function of Θ. By the inequality (12), for all $x \geqslant D_2$, we have

$$I_2 = E\left[\Pr(\Theta X > x, x/D_2 > \Theta \geqslant 1|\Theta)\right] = \int_1^{x/D_2} \Pr\left(X > \frac{x}{t}\right)dV(t) = \Pr(X > x)\int_1^{x/D_2} \frac{\Pr(X > \frac{x}{t})}{\Pr(X > x)}\,dV(t) \leqslant C_2\overline{F}(x)E\Theta^{p_2}.$$

Similarly, using the inequality (11), for all $x \geqslant D_1$,

$$I_3 = E\left[\Pr(\Theta X > x, \Theta < 1|\Theta)\right] \leqslant C_1^{-1}\overline{F}(x)E\Theta^{p_1}.$$

So let $C = \max(C_2, 1/C_1, C_2/\overline{F}(D_2))$, for all $x \geqslant \max(D_1, D_2)$, (17) holds. $\square$

Remark 3. Clearly, under the conditions of the lemma, for the non-identically distributed $\{X_i,\ i = 1, 2, \ldots\}$ that we study, by (11), (12) and assumption (A2), for any fixed p_1, p_2 satisfying $0 < p_1 < \mathbb{J}_G^- \leqslant \mathbb{J}_G^+ < p_2 < \infty$, for sufficiently large x, there exists one constant C such that:

$$\sum_{i=1}^{\infty} \Pr(\Theta_i X_i > x) \leqslant C\overline{G}(x) \sum_{i=1}^{\infty} E\Theta_i^{p_1} \vee E\Theta_i^{p_2}.$$

When (14) holds, for large enough i, we have $E\Theta_i^{p_1} \vee E\Theta_i^{p_2} \leqslant (E\Theta_i^{p_1})^{\frac{1}{p_2}} \vee (E\Theta_i^{p_2})^{\frac{1}{p_2}}$. Thus the sum on the right-hand side is bounded above under the conditions of Theorem 2.

Lemma 2. *Let X and Θ be two independent random variables. X is distributed by $F \in \mathcal{D}$ and Θ is nonnegative. Then for any fixed $p > \mathbb{J}_F^+$, there exists a positive constant C such that for all $x > 0$,*

$$E\big(\Theta X^+\big)^p I(\Theta X \leqslant x) \leqslant C \Pr(\Theta X > x) x^p.$$

Proof. By Theorem 3.3(iv) of Cline and Samorodnitsky [7] or Lemma 3.9 in Tang and Tsitsiashvili [18], the distribution function H of ΘX belongs to the class $\mathcal{D}$ and $\mathbb{J}_H^\pm = \mathbb{J}_F^\pm$. Then by (12), for every $\widetilde{p} \in (\mathbb{J}_F^+, p)$, there exist positive constants C_2 and D_2 such that, uniformly for all $x \geqslant y \geqslant D_2$,

$$\frac{\overline{H}(y)}{\overline{H}(x)} \leqslant C_2 \left(\frac{x}{y}\right)^{\widetilde{p}}.$$

Thus, when $x \leqslant D_2$,

$$E\big(\Theta X^+\big)^p I_{(\Theta X \leqslant x)} \leqslant x^p \leqslant x^p \frac{\overline{H}(x)}{\overline{H}(D_2)} \leqslant C x^p \overline{H}(x).$$

When $x > D_2$,

$$E\big(\Theta X^+\big)^p I_{(\Theta X \leqslant x)} \leqslant \int_0^x \overline{H}(t)\, dt^p \leqslant \left(\int_0^{D_2} + \int_{D_2}^{x}\right) \overline{H}(t)\, dt^p \leqslant C x^p \overline{H}(x) + \int_{D_2}^{x} \overline{H}(x) C_2 \left(\frac{x}{t}\right)^{\widetilde{p}} dt^p \leqslant C x^p \overline{H}(x).$$

Obviously, C only relates to F and p. $\square$

Remark 4. By (12), for the $\{X_i,\ i = 1, 2, \ldots\}$ we study, we can choose one common C for all $i \geqslant 1$ such that for any fixed $p > \mathbb{J}_G^+$ and all $x > 0$:

$$E\big(\Theta_i X_i^+\big)^p I(\Theta_i X_i \leqslant x) \leqslant C \Pr(\Theta_i X_i > x) x^p.$$

4.2. Proof of the main results

Proof. Mimicking the proof of the corresponding theorem in Chen and Yuen [6], one can prove Theorem 1 easily. We only prove Theorem 2.

First let's see (15). On one hand, for any fixed and sufficiently large m, by Theorem 1 and Lemma 1, we have

$$\begin{aligned}
&\liminf_{x\to\infty} \frac{\Pr(\max_{1\leqslant n<\infty} \sum_{i=1}^{n} \Theta_i X_i > x)}{\sum_{i=1}^{\infty} \Pr(\Theta_i X_i > x)} \\
&\quad \geqslant \liminf_{x\to\infty} \frac{\Pr(\max_{1\leqslant n\leqslant m} \sum_{i=1}^{n} \Theta_i X_i > x)}{\sum_{i=1}^{m} \Pr(\Theta_i X_i > x)} \cdot \liminf_{x\to\infty} \frac{\sum_{i=1}^{m} \Pr(\Theta_i X_i > x)}{\sum_{i=1}^{\infty} \Pr(\Theta_i X_i > x)} \\
&\quad \geqslant L \cdot \left(1 - \limsup_{x\to\infty} \frac{\sum_{i=m+1}^{\infty} \Pr(\Theta_i X_i > x)}{\sum_{i=1}^{\infty} \Pr(\Theta_i X_i > x)}\right) \geqslant L \cdot \left(1 - C\left(\sum_{i=m+1}^{\infty} E\Theta_i^{p_1} \vee E\Theta_i^{p_2}\right)\right).
\end{aligned}$$

The last inequality is due to $\Pr(\Theta_i X_i > x) \asymp \Pr(X_i > x)$ which is by Theorem 3.3 of Cline and Samorodnitsky [7]. Letting $m \to \infty$, we get

$$\Pr\left(\max_{1\leqslant n<\infty} \sum_{i=1}^{n} \Theta_i X_i > x\right) \gtrsim L \sum_{i=1}^{\infty} \Pr(\Theta_i X_i > x). \tag{18}$$

On the other hand, for any m such that $\forall i \geqslant m$, $E\Theta_i^{p_1} \vee E\Theta_i^{p_2} < 1$ and any $0 < \nu < 1$,

$$\begin{aligned}
&\limsup_{x\to\infty} \frac{\Pr(\max_{1\leqslant n<\infty} \sum_{i=1}^n \Theta_i X_i > x)}{\sum_{i=1}^\infty \Pr(\Theta_i X_i > x)} \\
&\quad \leqslant \limsup_{x\to\infty} \frac{\Pr(\max_{1\leqslant n\leqslant m} \sum_{i=1}^n \Theta_i X_i > (1-\nu)x)}{\sum_{i=1}^\infty \Pr(\Theta_i X_i > x)} + \limsup_{x\to\infty} \frac{\Pr(\sum_{i=m+1}^\infty \Theta_i X_i^+ > \nu x)}{\sum_{i=1}^\infty \Pr(\Theta_i X_i > x)} =: I_1 + I_2,
\end{aligned}$$

$$\begin{aligned}
I_1 &\leqslant \limsup_{x\to\infty} \frac{\Pr(\max_{1\leqslant n\leqslant m} \sum_{i=1}^n \Theta_i X_i > (1-\nu)x)}{\sum_{i=1}^m \Pr(\Theta_i X_i > (1-\nu)x)} \cdot \limsup_{x\to\infty} \frac{\sum_{i=1}^m \Pr(\Theta_i X_i > (1-\nu)x)}{\sum_{i=1}^m \Pr(\Theta_i X_i > x)} \\
&\quad \cdot \limsup_{x\to\infty} \frac{\sum_{i=1}^m \Pr(\Theta_i X_i > x)}{\sum_{i=1}^\infty \Pr(\Theta_i X_i > x)} \\
&\leqslant L^{-1} \cdot \bigvee_{1\leqslant i\leqslant m} \overline{F}_i^*(1-\nu).
\end{aligned}$$

Now let's see I_2. By Lemma 1,

$$\begin{aligned}
\Pr\Biggl(\sum_{i=m+1}^\infty \Theta_i X_i^+ > x\Biggr) &\leqslant \sum_{i=m+1}^\infty \Pr\bigl(\Theta_i X_i^+ > x\bigr) + \Pr\Biggl(\sum_{i=m+1}^\infty \Theta_i X_i^+ I_{(\Theta_i X_i^+ \leqslant x)} > x\Biggr) \\
&\leqslant C\overline{G}(x) \sum_{i=m+1}^\infty \bigl(E\Theta_i^{p_1} \vee E\Theta_i^{p_2}\bigr) + x^{-p_2} E\Biggl(\sum_{i=m+1}^\infty \Theta_i X_i^+ I_{(\Theta_i X_i^+ \leqslant x)}\Biggr)^{p_2} \\
&=: C\overline{G}(x) \sum_{i=m+1}^\infty \bigl(E\Theta_i^{p_1} \vee E\Theta_i^{p_2}\bigr) + J_1.
\end{aligned}$$

If $0 < \mathbb{J}_F^+ < 1$, apply the inequality: $|a+b|^r \leqslant |a|^r + |b|^r$ ($0 < r < 1$, $\forall a, b$), Lemmas 2 and 1,

$$\begin{aligned}
J_1 &\leqslant x^{-p_2} \sum_{i=m+1}^\infty E\bigl[\Theta_i X_i^+ I_{(\Theta_i X_i \leqslant x)}\bigr]^{p_2} \leqslant x^{-p_2} \sum_{i=m+1}^\infty C \Pr(\Theta_i X_i > x) x^{p_2} = C \sum_{i=m+1}^\infty \Pr(\Theta_i X_i > x) \\
&\leqslant C\overline{G}(x) \sum_{i=m+1}^\infty \bigl(E\Theta_i^{p_1} \vee E\Theta_i^{p_2}\bigr).
\end{aligned}$$

So we have

$$\Pr\Biggl(\sum_{i=m+1}^\infty \Theta_i X_i^+ > x\Biggr) \leqslant C\overline{G}(x) \sum_{i=m+1}^\infty \bigl(E\Theta_i^{p_1} \vee E\Theta_i^{p_2}\bigr).$$

If $\mathbb{J}_F^+ > 1$, apply the Minkowski's inequality, Lemmas 2 and 1,

$$\begin{aligned}
J_1 &\leqslant x^{-p_2} \Biggl(\sum_{i=m+1}^\infty \bigl(E\bigl(\Theta_i X_i^+\bigr)^{p_2} I_{(\Theta_i X_i \leqslant x)}\bigr)^{\frac{1}{p_2}}\Biggr)^{p_2} \leqslant C \Biggl(\sum_{i=m+1}^\infty \bigl(\Pr\bigl(\Theta_i X_i^+ > x\bigr)\bigr)^{\frac{1}{p_2}}\Biggr)^{p_2} \\
&\leqslant C\overline{G}(x) \Biggl(\sum_{i=m+1}^\infty \bigl(E\Theta_i^{p_1}\bigr)^{\frac{1}{p_2}} \vee \bigl(E\Theta_i^{p_2}\bigr)^{\frac{1}{p_2}}\Biggr)^{p_2}.
\end{aligned}$$

So we get

$$\Pr\Biggl(\sum_{i=m+1}^\infty \Theta_i X_i^+ > x\Biggr) \leqslant C\overline{G}(x) \Biggl(\sum_{i=m+1}^\infty \bigl(E\Theta_i^{p_1} \vee E\Theta_i^{p_2}\bigr) + \Biggl(\sum_{i=m+1}^\infty \bigl(E\Theta_i^{p_1} \vee E\Theta_i^{p_2}\bigr)^{\frac{1}{p_2}}\Biggr)^{p_2}\Biggr).$$

Since $\Pr(\Theta_1 X_1 > x) \asymp \Pr(X_1 > x)$, by assumption (A2) we have

$$I_2 \leqslant C \cdot \overline{F}_1^*(\nu) \cdot g(m+1),$$

where

$$g_(m+1) = \begin{cases} \sum_{i=m+1}^{\infty}(E\Theta_i^{p_1} \vee E\Theta_i^{p_2}) & \text{if (13) holds,} \\ \sum_{i=m+1}^{\infty}(E\Theta_i^{p_1} \vee E\Theta_i^{p_2}) + (\sum_{i=m+1}^{\infty}(E\Theta_i^{p_1})^{\frac{1}{p_2}} \vee (E\Theta_i^{p_2})^{\frac{1}{p_2}})^{p_2} & \text{if (14) holds.} \end{cases}$$

Thus,

$$\limsup_{x\to\infty} \frac{\Pr(\max_{1\leqslant n<\infty}\sum_{i=1}^{n}\Theta_i X_i > x)}{\sum_{i=1}^{\infty}\Pr(\Theta_i X_i > x)} \leqslant L^{-1} \bigvee_{1\leqslant i\leqslant m} \overline{F}_i^*(1-\nu) + C\overline{F}_1^*(\nu)g(m+1).$$

Letting $m \to \infty$ and $\nu \searrow 0$, by (13), (14) and Remark 3, we have

$$\Pr\left(\max_{1\leqslant n<\infty}\sum_{i=1}^{n}\Theta_i X_i > x\right) \lesssim L^{-2}\sum_{i=1}^{\infty}\Pr(\Theta_i X_i > x). \tag{19}$$

Consequently, combining (18) and (19), we obtain (15). The proof of (16) can be given straightforwardly by going along the same lines as above. □

Acknowledgments

This work was supported by the National Natural Science Foundation (Grant Nos. 10801124, 10671188). The authors would like to thank the anonymous referee, Professors Qihe Tang and Yuebao Wang for their help.

References

[1] S. Aljančić, D. Arandelović, O-regularly varying functions, Publ. Inst. Math. (Beograd) (N.S.) 22 (36) (1977) 5–22.
[2] N.H. Bingham, C.M. Goldie, J.L. Teugels, Regular Variation, Cambridge University Press, Cambridge, 1987.
[3] J. Cai, Q. Tang, On max-sum equivalence and convolution closure of heavy-tailed distributions and their applications, J. Appl. Probab. 41 (1) (2004) 117–130.
[4] Y. Chen, K.W. Ng, X. Xie, On the maximum of randomly weighted sums with regularly varying tails, Statist. Probab. Lett. 76 (2006) 971–975.
[5] Y. Chen, C. Su, Finite time ruin probability with heavy-tailed insurance and financial risks, Statist. Probab. Lett. 76 (2006) 1812–1820.
[6] Y. Chen, K.C. Yuen, Sums of pairwise quasi-asymptotically independent random variables with consistent variation, Stoch. Models 25 (2009) 76–89.
[7] D.B.H. Cline, G. Samorodnitsky, Subexponentiality of the product of independent random variables, Stochastic Process. Appl. 49 (1994) 75–98.
[8] P. Embrechts, C. Klüppelberg, T. Mikosch, Modelling Extremal Events for Insurance and Finance, Springer-Verlag, Berlin, 1997.
[9] Q. Gao, Y. Wang, Randomly weighted sums with dominatedly varying-tailed increments and application to risk theory, J. Korean Statist. Soc. 39 (3) (2010) 305–314.
[10] J.L. Geluk, C.G. De Vries, Weighted sums of subexponential random variables and asymptotic dependence between returns on reinsurance equities, Insurance Math. Econom. 38 (2006) 39–56.
[11] J. Geluk, Q. Tang, Asymptotic tail probabilities of sums of dependent subexponential random variables, J. Theoret. Probab. 22 (2009) 871–882.
[12] M.J. Goovaerts, R. Kaas, R.J.A. Laeven, Q. Tang, R. Vernic, The tail probability of discounted sums of Pareto-like losses in insurance, Scand. Actuar. J. (2005) 446–461.
[13] H. Nyrhinen, On the ruin probability in a general economic environment, Stochastic Process. Appl. 83 (1999) 319–330.
[14] H. Nyrhinen, Finite and infinite time ruin probabilities in a stochastic economic environment, Stochastic Process. Appl. 92 (2001) 265–285.
[15] S.I. Resnick, E. Willekens, Moving averages with random coefficients and random coefficients autoregressive models, Commun. Statist. Stochastic Models 7 (1991) 511–525.
[16] Q. Tang, Asymptotics for the finite time ruin probability in the renewal model with consistent variation, Stoch. Models 20 (2004) 281–297.
[17] Q. Tang, Asymptotic ruin probabilities of the renewal model with constant interest force and regular variation, Scand. Actuar. J. 1 (2005) 1–5.
[18] Q. Tang, G. Tsitsiashvili, Precise estimates for the ruin probability in finite horizon in a discrete-time model with heavy-tailed insurance and financial risks, Stochastic Process. Appl. 108 (2003) 299–325.
[19] Q. Tang, G. Tsitsiashvili, Randomly weighted sums of subexponential random variables with application to ruin theory, Extremes 6 (2003) 171–188.
[20] Q. Tang, G. Tsitsashvili, Finite and infinite time ruin probabilities in the presence of stochastic return on investments, Adv. in Appl. Probab. 36 (2004) 1278–1299.
[21] D. Wang, C. Su, Y. Zeng, Uniform estimate for maximum of randomly weighted sums with applications to insurance risk theory, Sci. China Ser. A 48 (2005) 1379–1394.
[22] D. Wang, Q. Tang, Tail probabilities of randomly weighted sums of random variables with dominated variation, Stoch. Models 22 (2006) 253–272.
[23] Y. Yang, Y. Wang, Asymptotics for ruin probability of some negatively dependent risk models with a constant interest rate and dominatedly-varying-tailed claims, Statist. Probab. Lett. 80 (2009) 143–154.
[24] Y. Zhang, X. Shen, C. Weng, Approximation of the tail probability of randomly weighted sums and applications, Stochastic Process. Appl. 119 (2009) 655–675.